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A Necessary Condition for Local Solvability for a Class of Operators with Double Characteristics

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1. INTRODUCTION AND STATEMENT OF THE RESULT

In this paper we prove that condition sub($\mathscr{P}$) (see below), stated as a conjecture in the introduction of [5], is necessary for local solvability at a point for certain pseudodifferential operators with double involutive characteristics. The main novelty of this result is not in the method of proof, which follows the pattern originally presented by Hörmander in [2] (we shall rely heavily on Hörmander [3]), but rather the fact that the operator is not of principal type. That the condition is on the subprincipal symbol should not come as a surprise since many results in the literature on solvability and well posedness of the Cauchy problem for operators with characteristics of multiplicity higher than 1 involve conditions on lower order terms of the full symbol of the operator.

We shall work with a classical, properly supported operator P on an open set X in $\mathbb{R}^n$ whose principal symbol p is real and factorizes microlocally, i.e., near any point in $T^*X\backslash 0$, $p = p_1 p_2$ with p_j real valued, C^∞ and homogeneous. We assume the doubly characteristic set $\sum = \{\nu \in T^*X\backslash 0 \mid p(\nu) = dp(\nu) = 0\}$ to be an involutive submanifold of codimension 2 and that at points in Σ, H_{p_1}, H_{p_2} and the cone direction are independent.

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We shall prove that a necessary condition for microlocal solvability at a point $v_0 \in \Sigma$, for the class of operators just described, is

$$\mathrm{Sub}(\mathscr{P}) \qquad \operatorname{Im} \sigma_{\mathrm{sub}}(\mathscr{P}) \text{ does not change sign at } v_0 \text{ along either the bicharacteristics of } p_1 \text{ or those of } p_2.$$

As usual, $\sigma_{\mathrm{sub}}(\mathscr{P})$ denotes the subprincipal symbol of P.

One can easily prove that the bicharacteristic curves involved in the condition are changed only by a reparametrization if the factorization $p = p_1 p_2$ is changed. Thus if P is regarded as an operator on 1/2-densities the condition is invariant under conjugation of P by elliptic Fourier integral operators.

We take over from Hörmander [3] the following

1.1. Definition. Let $K \subset S^*X$ be a compact subset of the cosphere bundle of X. P is locally solvable at K if there exists an integer N such that for every $f \in H^{\mathrm{loc}}_{(N)}(X)$ there exists $u \in \mathscr{D}'(X)$ for which $Pu - f$ is smooth near K.

Our theorem can now be stated as follows:

1.2. Theorem. *Condition* sub($\mathscr{P}$) *is necessary in order that P be locally solvable at $v_0 \in \Sigma$.*

To motivate the proof, we want to explain how the conjecture that condition sub($\mathscr{P}$) is necessary originated in [5]. For this, consider the operator

$$P = D_{x_1} D_{x_2} + B(x, D_x), B \in L^1_{cl}(\mathbb{R}^n), \qquad n \geqslant 3 \tag{1.1}$$

which is a microllocal model for the class of operators considered here (see [6]). Let b be the principal symbol of B. In the case $\operatorname{Im} b \neq 0$ at Σ, we constructed in [5] a solution of $Pu = \delta$ (δ is the δ-function at 0) of the form

$$u = \int e^{i((x'-s)\xi' + x''\xi' + s \cdot t|\xi|)} \frac{a(x, s, t, \xi)}{t_1 t_2 + \beta(x, s, \xi)} \, dt ds d\xi, \tag{1.2}$$

where $x = (x', x'') \in \mathbb{R}^2 \times \mathbb{R}^{n-2}$, $\xi = (\xi', \xi'') \in \mathbb{R}^2 \times \mathbb{R}^{n-2}$, $a \in S^m(\mathbb{R}^n \times \mathbb{R}^4 \times \mathbb{R}^n \backslash 0)$ is a symbol with compact support in (s, t) and $\beta(x, s, \xi)$ is a homogeneous function of degree -1 related to b, as indicated below, in such a way that $\operatorname{sgn} \operatorname{Im} b = \operatorname{sgn} \operatorname{Im} \beta$.

For u as in (1.2) we have, if $\operatorname{Im} b < 0$,

$$WF(u) \subset \Lambda_0 \cup \Lambda_1^+ \cup \Lambda_2^+.$$

where $\Lambda_0 = \{(0;\xi) \in T^*\mathbb{R}^n \backslash 0\}$, $\Lambda_1^+ = \{(x,\xi) \in N^*(x_2 = x'' = 0) | x_1\xi_2 \geqslant 0\}$ and $\Lambda_2^+ = \{(x,\xi) \in N^*(x_1 = x'' = 0) | x_2\xi_1 \geqslant 0\}$. The symbol of u on $\Lambda_1 \backslash \{\xi_2 = 0\}$ behaves essentially like

$$w_1 = ie^{-ix_1\beta(x_1,0,x'',(x_1,0),0,\xi_2,\xi'')|\xi|^2/\xi_2}(H(x_1)\,H(\xi_2) - H(-x_1)\,H(-\xi_2))/\xi_2, \quad (1.3)$$

where H is the Heaviside function. In order to actually get a solution of $Pu = \delta$ one chooses β so that $\beta(x_1, 0, x'', (x_1, 0), 0, \xi_2, \xi'')\,|\xi|^2/\xi_2 = \int_0^{x_1} b(r, 0, x'', 0, \xi_2, \xi'')\,dr$. Thus w_1 satisfies $(-i\,\xi_2 H_{\xi_1} + b)\,w_1 = \delta(x_1)$ (the first transport equation on Λ_1) on Λ_1 and this was the reason for defining distributions like (1.2) in [5].

Actually, on $\xi_2 \neq 0$ the solutions of $(-i\xi_2 H_{\xi_1} + b)\,w_1 = \delta(x_1)$ are all of the form

$$w = e^{-ix_1\beta|\xi|^2/\xi_2}(c_1 H(x_1) + c_2 H(-x_1))/\xi_2$$

for appropriate c_1 and c_2 but if b has a change of sign along the integral curve of H_{ξ_1} through a point v_0 in $x_1 = 0$, $\xi_2 = 0$, then there are no distributional solutions of $(-i\xi_2 H_{\xi_1} + b)\,w = \delta(x_1)$ in a neighborhood of v_0 which agree with w_1 on $\xi_2 \neq 0$ due of course to the behavior of the exponential function in w.

2. Proof of the Theorem

Since both condition sub($\mathscr{P}$) and solvability at v_0 are preserved under conjugation of P by elliptic FIOs, we may assume

$$P = D_{x_1} D_{x_2} + B(x, D_x) \quad (2.1)$$

with $B \in L^1_{cl}(\mathbb{R}^n)$ and $v_0 = (0; 0,..., 0, 1) \in T^*\mathbb{R}^n$. Now, if P is solvable at v_0 and W is a bounded neighborhood of 0 in $\mathbb{R}^n$ one can find an integer m and a pseudodifferential operator A which is smoothing near v_0, such that for all $v \in C_0^\infty(W)$

$$\|v\|_{-m} \leqslant C(\|P^*v\|_N + \|v\|_{-m-n} + \|Av\|_0) \quad (2.2)$$

(by Proposition 2.5 of [3]). We shall then assume that P violates condition sub($\mathscr{P}$) at v_0 and construct asymptotic solutions to $P^*v_\tau = 0$, $v_\tau \in C_0^\infty(W)$ of the form $e^{i\tau g}a$. Let $b = \sigma(B^*) = \sigma_{\text{sub}}(P^*)$. Without loss of generality we may assume $\operatorname{Im} b$ changes sign at $t = 0$ along the integral curve $t \to \gamma(t) = (t, 0..., 0; 0,..., 1)$ of H_{ξ_1} through v_0.

The first thing to do, following the ideas mentioned in the previous section is to get b on $\{\xi_1 = 0\}$ in the first term of the asymptotic expansion in τ of $u_\tau = P(v_\tau)$. Thus we take g independent of x_1. Second, we want to get as a main contribution from b in the asymptotic expansion of u_τ, $b(x_1, 0; 0, 1)$. In order to do that, we make the asymptotic change of variables (as in Ivrii and Petkov [4]).

$$y_j = \tau^{s_j} x_j, \qquad j = 1, \dots, n \tag{2.3}$$

with $s_1 = 0 < s_2 < s_3 = \dots = s_n < 1$, $2 s_2 = s_n$. (We shall take $s_2 = \frac{1}{3}$, and choose $g(y) = y, \eta$, where $\eta = (0, \eta_2 0, \dots, 1) \in \mathbb{R}^n$, $\eta_2 \neq 0$. The reason for making the change of variables (2.3) is to get the main contribution from b at $x_2 = \cdots = x_n = 0$ since $x_j = y_j / \tau^{s_j} \to 0$ as $\tau \to \infty$. Also, we weigh x_2 lower than $x_j (j > 2)$ to be able to approach $\{\xi_2 = 0\}$, where the change of sign of $\operatorname{Im} b$ occurs, as $\tau \to \infty$. Under the change (2.3), P is transformed into

$$P_\tau(y, D_y) = \tau^{s_2} D_{y_1} D_{y_2} + B(\tau^{-s} y, \tau^s D_y),$$

where $\tau^{-s} y$ means $(\tau^{-s_1} y_1, \dots, \tau^{-s_n} y_n)$, and similarly $\tau^s D_y$. Let $\phi\colon \mathbb{R}^n \to \mathbb{R}^n$ be the map $\phi(x) = \tau^s x$, so that $P_\tau = \phi^{-1*} P \phi^*$. Then if P is solvable at v_0 we have from (2.2)

$$\|\phi^* v\|_{-m} \leqslant C(\|\phi^* P_\tau^* v\|_N + \|\phi^* v\|_{-m-N} + \|\phi^* A_\tau v\|_0)$$

if $v \in C_0^\infty(W)$, where $A_\tau = \phi^{-1*} A \phi^*$. In order to contradict this bound, we are going to construct $v_\tau(y) = e^{i\tau g + i\tau^{1/3}\beta} \sum_{j=1}^{M} \tau^{-j/3} \phi_j(y, \tau)$ with the ϕ_j in a bounded subset in $C_0^\infty(W)$ and $\beta = \beta(y, \tau)$ in a bounded subset in $C^\infty(W)$ and $\operatorname{Im} \beta \geqslant 0$ in W. Thus we need the following bounds which we obtain from Lemma 6.1 of [3].

2.1. Lemma. *Let v_τ be as above. Then*

(1) $\|v_\tau\|_{-m} \leqslant C_m \tau^{-m}$ *for all* $m > 0$.

(2) *If* $\phi_0(y_0) \neq 0$ *for some* $y_0 \in W$, *where* $\operatorname{Im} \beta(y_0) = 0$, *then* $\|v_\tau\|_{-m} \geqslant C \tau^{-m-n/2}$.

Since $\|\phi^* A_\tau v_\tau\|_0$ is very small if the support of the ϕ_j is close enough to 0 we only need to find v_τ so that $\|P_\tau^* v_\tau\|_N \leqslant C \tau^{-N-n/2-1}$. To do this we observe that by standard asymptotic expansion results (see [1], for instance) we have, if $a = e^{i\tau^{1/3}\beta} \phi$ with β and ϕ as ϕ_j above and $\sigma_l(P^*) \sim \sum p_{2-k}$, p_j homogeneous, that

$$\begin{aligned} P_\tau^*(y, D_y)\, e^{i\tau g}\, a = e^{i\tau g}\Big(&\tau^{4/3} \eta_2 D_{y_1} a + \tau^{5/3} b(\tau^{-s} y, \tau^{s-2/3} \eta)\, a \\ &+ \sum_{j=0}^{l} \tau^{(2-j)/3} L_j a + R_l(a)\Big), \end{aligned}$$

where $L_j = \sum_{|\alpha| \leqslant l} q_{\alpha, j}(\tau^{-s} y, \tau^{s-2/3} \eta)\, D_y^\alpha$ with $q_{\alpha, j}$ linear combination of derivatives of the p_ν, $q_{\alpha,j} = 0$ if $|\alpha| > 2 + (j-2)/5$, and the remainder satisfies $\|R_l(a)\, e^{i\tau g}\|_N \leqslant C_l \tau^{M_l}$, where $M_l \to -\infty$ if $l \to \infty$.

Thus the first transport equation we have to solve is

$$\tau^{4/3}\, \eta_2 D_{y_1} a_0 + \tau^{5/3}\, b(\tau^{-s} y, \tau^2 \eta)\, a_0 = 0.$$

Its solution of course is $a_0 = e^{i\tau^{1/3}\beta}$ with $\beta = -\int_c^{y_1} b(\tau^{-s}(r, y');\, \tau^{s-2/3}\eta)\, dt/\eta_2$. We shall choose c and the sign of η_2 in such a way that $\operatorname{Im}\beta \geqslant 0$ if τ is large. We have

$$\beta = \int_c^{y_1} b(t, 0; 0, \ldots, 0, 1)\, dt/\eta_2^0 + \tau^{-1/3} \beta_1(y, \tau)$$

with β_1 in a bounded set in $C^\infty(W)$. Since $\operatorname{Im} b$ changes sign at ν_0 along γ we may find $t_0 < 0 < t_1$ arbitrarily close to 0 such that, say, $\operatorname{Im} b(\gamma(t_0)) < 0$ and $\operatorname{Im} b(\gamma(t_1)) > 0$. This implies that any primitive of $\operatorname{Im} b \circ \gamma$ has a strict minimum in $[t_0, t_1]$ at some interior point c which we then use as the lower limit of integration to define β. We now choose the sign of η_2 so that $\operatorname{Im}\beta \geqslant 0$ (thus $\operatorname{sgn}\, \eta_2 = 1$). It follows that outside a compact interval $I \subset (t_0, t_1)$ $\operatorname{Im}\beta$ is bounded from below by a positive constant on all of W if τ is sufficiently large.

Now let $\phi \in C_0^\infty(t_0, t_1)$ and $\psi \in C_0^\infty(\mathbb{R}^{n-1})$ be such that $\phi = 1$ in a neighborhood of I and $\psi(0) \neq 0$. We may assume t_0 and t_1 as close to 0 as we wish and take ψ with support so small that $\operatorname{supp}(\phi(y_1)\, \psi(y_2, \ldots, y_n)) \subset W$. Then $P_\tau^*(e^{i\tau g} a_0 \phi\psi) = \sum_{j=0}^{l} \tau^{(2-j)/3} L_j(a_0 \phi\psi)$ plus an error $e_{l,0}$ such that $\|e_{l,0}\|_N \leqslant C_l \tau^{-M_l}$. We now proceed by induction and obtain v_τ as desired solving successive inhomogeneous transport equations to obtain the lower order terms in the expansion of v_τ, following the usual procedure. Thus the theorem is proved.

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